entering into the solution for the plane jet is expressed through the integral characteristics of the swirled fan jet in a way analogous to (2.8) and (2.10):

$$N = \frac{E}{2\pi x_0}\sqrt{1+\lambda_0^2},\ \lambda_0 = \frac{M}{Ex_0},$$

where (see [2])

$$E = 2\pi\rho x_0^2 \int_0^\infty u^2(x_0, y)\left[\int_0^y u(x_0, y)\,dy\right]dy;$$

$$M = 2\pi\rho x^3 \int_0^\infty uw\left(\int_0^y u\,dy\right)dy = \text{const}.$$

In conclusion, we note that all the solutions found earlier for free swirled fan jets are obtained as particular cases from the results of the present work.

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STABILITY OF POISEUILLE FLOW IN AN ELASTIC CHANNEL

O. Yu. Tsvelodub

UDC 532.5

The stability of a laminar boundary layer at a surface of the membrane type has been analyzed in [1, 2], while the stability of Poiseuille flow between membranes are analyzed in [3, 4]. Walls with a linear relationship between the perturbation of the pressure and the normal deformation of the surface were taken as the channel boundaries in [5]. The stability of the profile V = sin y (0 ≤ y ≤) was analyzed numerically in [6]. The stability of Poiseuille flow in a channel whose walls are elastic plates is studied in the present report. In contrast to [3, 5, 6], pulsations of the friction at the channel walls are taken into account along with pressure pulsations, just as in [4]. It is shown that a significant reorganization of the regions of instability occurs when they are allowed for. A region of instability is found which exists for any finite Reynolds number.

A stream whose velocity profile is $V \equiv V_x = 1 - y^2$ in a channel with walls $y = \pm 1$ is analyzed (Fig. 1). For the normal and tangential displacements of the upper plate we have [7]

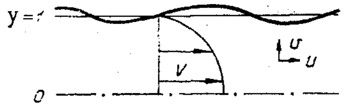

Fig. 1

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 75-80, September-October, 1977. Original article submitted September 16, 1976.

$$\rho_0 h\partial^2\tilde{\xi}/\partial t^2 = -Eh^3/[12(1-\sigma^2)]\partial^4\tilde{\xi}/\partial x^4 + p - \tau_{yy}; \quad (1)$$

$$\rho_0 h\partial^2\tilde{\beta}/\partial t^2 = Eh/(1-\sigma^2)\partial^2\tilde{\beta}/\partial x^2 + \tau_{xy}, \quad (2)$$

where $\tilde{\xi}$ and $\tilde{\beta}$ are the normal and tangential displacements, respectively, of a point of the plate from the position in a state of rest; ρ_0 and h are the density and thickness of the plate; E is Young's modulus; σ is the Poisson ratio; $p - \tau_{yy}$ and τ_{xy} are the normal and tangential stresses caused by the puslations in the velocities of the perturbing motion. The equations are written in dimensionless form and the corresponding quantities are normalized to the half-width of the channel, the density of the fluid, and the velocity V_m at the axis of the channel. In the linear approximation the connection between $\tilde{\xi}$ and $\tilde{\beta}$ and the components of the perturbation velocity has the form

$$\partial\tilde{\xi}/\partial t = \tilde{v}; \quad (3)$$

$$\partial\tilde{\beta}/\partial t = \tilde{u} + V'\tilde{\xi}, \; y = \pm 1. \quad (4)$$

All the perturbed quantities are sought in the form

$$\tilde{A}(x, y, t) = A(y)\exp[i\alpha(x - ct)], \quad (5)$$

where α is the wave number; $c = c_r + ic_i$ is the complex velocity.

Neglecting the nonlinear terms in the equations of motion of the fluid and using (1)-(5) for the amplitude of the stream function $\varphi(y)$ of the perturbing flow we obtain the Orr–Sommerfeld equation

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha\,\mathrm{Re}\,\{(V - c)(\varphi'' - \alpha^2\varphi) - V''\varphi\}. \quad (6)$$

We write the boundary conditions at the upper plate as

$$\varphi''' = -i\alpha\,\mathrm{Re}\,\{(\rho_0 h\alpha^2 c - K\alpha^4/c)\varphi + c\varphi' + V'\varphi\} + 3\alpha^2\varphi'; \quad (7)$$

$$\varphi' + \varphi V'/c = -[\varphi'' + (\alpha^2 + V''/c)\varphi]/[i\alpha\,\mathrm{Re}\,(\rho_0 hc - K_1/c)], \; y = 1, \quad (8)$$

where $\mathrm{Re} = V_m L/\nu$, $K = Eh^3/[12(1-\sigma^2)]$, $K_1 - Eh/(1-\sigma^2)$. In the calculations E and h were taken as constant, by analogy with [4-6], and therefore in the figures presented below Re varies only by virtue of the variation of ν.

Since $K_1 >> K$ for the plates, in the majority of cases the longitudinal displacements can be neglected. Then the conditions at the boundary have the form

$$\varphi''' = -i\alpha\,\mathrm{Re}(\rho_0 h\alpha^2 c - K\alpha^4/c)\varphi + 3\alpha^2\varphi'; \quad (9)$$

$$\varphi' + \varphi V'/c = 0, \; y = 1. \quad (10)$$

If, in addition, one does not take τ_{yy} into account in (1) [2, 5, 6], then (9) is rewritten in the form

$$\varphi''' = -i\alpha\,\mathrm{Re}(\rho_0 h\alpha^2 c - K\alpha^4/c)\varphi + \alpha^2\varphi', \; y = 1. \quad (11)$$

Henceforth, we will be confined to the consideration of perturbations which are symmetric relative to φ. At the axis of the channel we have

$$\varphi'(0) = \varphi'''(0) = 0. \quad (12)$$

In the solution of the given boundary problem the main interest is in finding the neutral curves ($c_i = 0$) separating the regions of growing perturbations ($c_i > 0$) from the regions of damped perturbations ($c_i < 0$).

Let Re << 1. In this case, regardless of the form of the profile of the main flow, the solution of the boundary problem (6), (9), (10), (12) can be obtained in the form of a series with respect to Re:

$$\varphi = \varphi_0 + \mathrm{Re}\,\varphi_1 + \ldots; \quad (13)$$

$$c = c_0 + \mathrm{Re}\,c_1 + \ldots \quad (14)$$

Proceeding by analogy with [8], we substitute (13) and (14) into these equations, and, equating the coefficients of equal powers of Re, for the zeroth approximation we obtain

$$\varphi_0 = \mathrm{ch}\,\alpha y - y\,\mathrm{th}\,\alpha\,\mathrm{sh}\,\alpha y; \quad (15)$$

$$c_0 = V'(1)/\mathrm{sh}^2\alpha. \quad (16)$$

TABLE 1

Source	Re_*	α	c_r	c_i
[11]	5772,23	1,02041	0,26399	$-0{,}4\cdot 10^{-9}$
Present work	5772,2	1,02	0,26394	0

It follows from (16) that the axis Re = 0 is a neutral curve and, for small Re, perturbations propagate upstream ($c_0 < 0$). For an analysis of the stability of these perturbations one must consider the next term of the series (14). After transformations we obtain

$$c_1 = ic_0^2/(2\alpha^3V')\{(\alpha^2 + 0.5\alpha\,\mathrm{sh}\,2\alpha)\,L(c_0) - [\alpha^3/3 - (0.75V' + 0.25)\alpha]\,\mathrm{sh}\,2\alpha + \\ + \alpha^3V'\,\mathrm{cth}\,\alpha + (2.25 + 1.125\,\mathrm{sh}\,2\alpha/\alpha)\,\mathrm{sh}^2\alpha + (1.5V' + 1)\,\alpha^2 - 0.5\,\mathrm{sh}^2 2\alpha\}, \tag{17}$$

where $L(c_0) = \alpha^2(\rho_0 hc_0 - K\alpha^2/c_0)$.

An analysis of the expression (17) shows that c_1 is a purely imaginary quantity and an α_* exists such that

$$\mathrm{Im}\,(c_1)\begin{cases} > 0 & \text{for } \alpha < \alpha_*, \\ = 0 & \text{for } \alpha = \alpha_*, \\ < 0 & \text{for } \alpha > \alpha_*. \end{cases}$$

Thus, α_* is a branching point of the neutral curve. The appearance of growing perturbations at low Reynolds numbers is due to the presence of elastic boundaries: $\alpha_* \to 0$ at $K \to \infty$ (solid walls) and this unstable region disappears.

For Re << 1 one must also allow for tangential displacements and (7) and (8) must be used instead of (9) and (10). In this case φ_0 does not change and c_0 and c_1 have the form

$$c_0 = V''(1)/(\alpha\,\mathrm{sh}\,2\alpha - 2\alpha^2); \tag{18}$$

$$c_1 = ic_0^2/(\alpha^2V'')\{(0.625 - \mathrm{ch}^2\alpha)\,\mathrm{sh}\,2\alpha + (1.5\alpha - 2\alpha^3/3)\,\mathrm{sh}^2\alpha + \\ + 9\,\mathrm{sh}^2 2\alpha/(16\alpha) + 0.5\alpha + (0.375\alpha\,\mathrm{sh}^2 2\alpha + \alpha^3\,\mathrm{sh}^2\alpha + \\ + 0.25\alpha^2\,\mathrm{sh}\,2\alpha)\,c_0 + \alpha L(c_0)\,\mathrm{ch}^2\alpha + \alpha\,(V'/c_0 - \mathrm{sh}^2\alpha)\,(\alpha^2L_1(c_0) + c_0\,\mathrm{ch}^2\alpha)\}, \tag{19}$$

$$L_1(c_0) = \alpha^2(\rho_0 hc_0 - K_1/c_0).$$

For Poiseuille flow the character of the perturbations does not change qualitatively, although the value of α_* does become different.

As calculations show, for $K \gg 1$ in the first case the first two terms of the series (14) determine c accurately enough for $Re \le 0.1$–0.5, while in the second case they determine c accurately enough for $Re \leqslant 10^{-3}$. Since the neutral curves obtained with and without allowance for τ_{xy} practically coincide for the given K, even with $Re \simeq 0.05$–0.1, it is convenient to use Eqs. (16) and (17) for an approximate determination of the neutral curve for small but finite Re (0.1–0.5).

If τ_{yy} is not taken into account and (11) is used instead of (9), then the following expressions are obtained for φ_0, c_0, and c_1:

$$\varphi_0 = \mathrm{ch}\,\alpha y; \tag{20}$$

$$c_0 = -V''(1)\,\mathrm{cth}\,\alpha/\alpha; \tag{21}$$

$$c_1 = -ic_0^2/(\alpha V'\,\mathrm{sh}\,2\alpha)\{\alpha + 1.5\,\mathrm{sh}^2\alpha/\alpha + 1.5\,\mathrm{th}\,\alpha + (\alpha + 0.5\,\mathrm{sh}\,2\alpha)\,L(c_0)\}. \tag{22}$$

In comparing (20)-(22) with (15)-(18) it is seen that the behavior of the perturbations changes qualitatively. The region of instability also corresponds to small α, but in this case the perturbations propagate downstream ($c_0 > 0$).

Evidently τ_{yy} cannot be neglected for small Reynolds numbers. Using the results obtained for small Re with and without allowance for τ_{yy}, one is also able to find perturbations for which the form of the neutral curve essentially depends on whether (9) or (11) is taken as the boundary condition, even in the case of large Re when the viscous forces acting on the elastic boundary are usually neglected [3-5].

For finite Re the boundary problem (6), (9), (10), (12) was solved numerically by the determinant method [9]. The main results were reproduced using the differential trial-run

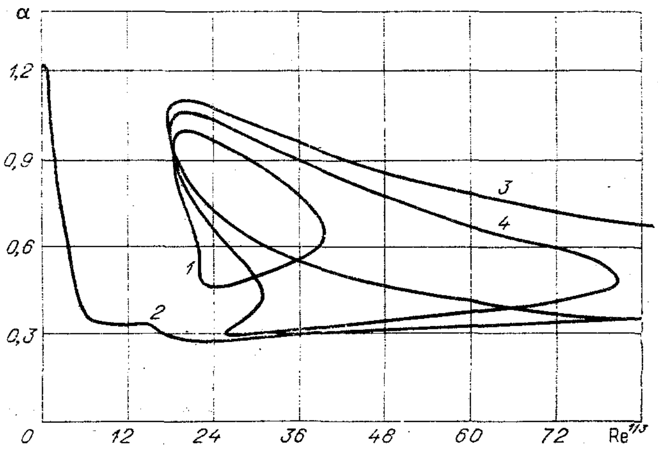

Fig. 2

method [10]. The difference in the wave numbers of the neutral perturbations calculated by these two methods hardly exceeded the assigned accuracy for the determination of α. The critical Reynolds number Re_* found for a channel with rigid walls agrees well with that presented in [11] (see Table 1).

For the variants presented below the inertia of the plates can be neglected and their other properties can be characterized by the one parameter K. The calculations show that with a decrease in K the region of instability of Tollmien-Schlichting waves (region I) shifts toward smaller wave numbers and larger Reynolds numbers. For small enough K this region is closed.

If $\varphi'|_{y=\pm 1} = 0$ is taken instead of (10) as the boundary condition, as is done in [5], then the behavior of Re_* changes qualitatively — Re_* also decreases with a decrease in K.

The perturbations, which move upstream for small Re, begin to propagate downstream at large enough Reynolds numbers. Therefore, at some α and Re their phase velocity is reduced to zero. It is seen from (5) that for such perturbations the departure of the points of the boundary from the unperturbed position will either increase (decrease) with time, if α lies in the region of instability (stability), or remain constant if α lies on a neutral curve. In the latter case the perturbed flow will be steady. The Reynolds number at which such a steady perturbation exists increases with an increase in K.

The region of growing perturbations, determined with the help of (15)-(17) for small Re, is called region II.

In Fig. 2 the neutral curves 1 and 2 bound regions I and II, respectively, for a plate with K = 1.39. The boundaries of region I for plates with K = ∞ and 3 (curve 3 and 4) are given for comparison. Region II decreases with an increase in K, the minimum distance between regions I and II also decreases, and at large enough K they merge.

If τ_{yy} is neglected and (11) is used instead of (9) then a new region of instability appears, region III, determined from (20)-(22) for small Re, while region II disappears. Therefore, in the case when regions I and II merge, the closure of region I occurs when τ_{yy} is neglected. But if these regions are separate then the location of region I and the phase

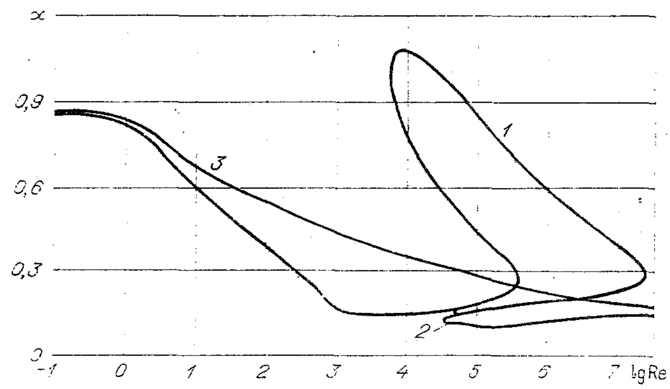

Fig. 3

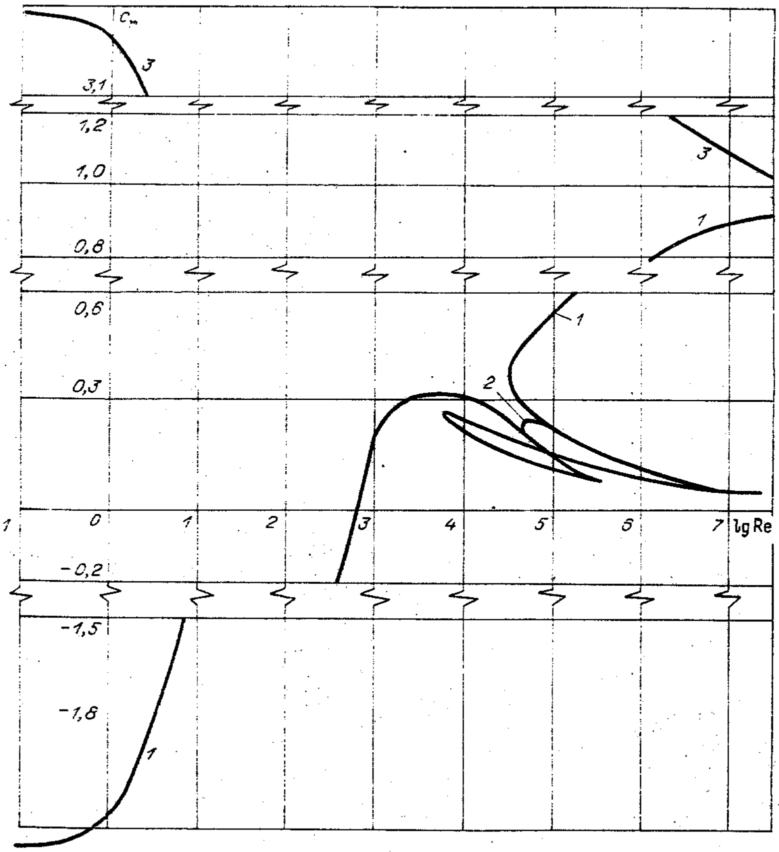

Fig. 4

velocity of the neutral perturbations at its boundary are hardly changed when (11) is used. For region III the velocity of propagation of the neutral perturbations is positive for all Re.

The neutral curves for a plate with K = 11.11 are given in Fig. 3. In this case regions I and II merge (curve 1), closure of region I along the line 2 occurs when τ_{yy} is neglected, and the curve 3 is the boundary of region III. The corresponding functions of the phase velocities are presented in Fig. 4.

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VISCOSITY OF A DILUTE SUSPENSION OF RIGID SPHERICAL PARTICLES IN A NON-NEWTONIAN FLUID

Yu. I. Shmakov and L. M. Shmakova

UDC 532.135

Consider the perturbations introduced by a rigid spherical particle of radius a suspended in a non-Newtonian fluid flow having a parallel velocity gradient

$$v_x = -(q/2)x,\ v_y = -(q/2)y,\ v_z = qz \tag{1}$$

and satisfying the Ostwald–Deville law

$$P = -pE + m(I/2)^{(n-1)/2}\dot{S}, \tag{2}$$

where v_x, v_y, v_z are the velocity components in a Cartesian coordinate system Oxyz with origin at the center of the particle; q is the constant; P is the stress tensor; $\dot{S}$ is the strain rate tensor with components $\dot{S}_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$, i, j = 1, 2, 3; I is the second invariant of the tensor $\dot{S}$; p is the pressure, E is the unit tensor, m is the consistency index; and n is the index of non-Newtonian behavior.

Transforming to a spherical coordinate system (r, θ, φ), we introduce the stream function $\psi(r, \theta)$, which is related to the velocity components by the expressions

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{3}$$

Now the equations of motion for a power-law fluid are written as follows, neglecting inertial forces (the generalized Reynolds number with respect to the particle is small):

$$\begin{aligned} \frac{\partial p}{\partial r} &= m\left(\frac{I}{2}\right)^{\frac{n-1}{2}}\left[-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}E^2\psi + \frac{n-1}{2}\left\{2\frac{\partial v_r}{\partial r}\frac{\partial \ln I}{\partial r} + \frac{1}{r}\left(r\frac{\partial}{\partial r}\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\frac{\partial \ln I}{\partial\theta}\right\}\right], \\ \frac{1}{r}\frac{\partial p}{\partial\theta} &= m\left(\frac{I}{2}\right)^{\frac{n-1}{2}}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial r}E^2\psi + \frac{n-1}{2}\left\{\left(r\frac{\partial}{\partial r}\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\frac{\partial \ln I}{\partial r} + \frac{1}{r^2}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right)\frac{\partial \ln I}{\partial\theta}\right\}\right], \end{aligned} \tag{4}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

and the boundary conditions for the problem assume the form

$$\begin{aligned} &v_\theta = v_r = 0 \quad \text{at} \quad r = a; \\ &v_r = (qr/2)(2\cos^2\theta - \sin^2\theta),\ v_\theta = -(3qr/2)\sin\theta\cos\theta \quad \text{as} \quad r \to \infty. \end{aligned} \tag{5}$$

Let $(n-1)/2 \ll 1$ (the dispersion medium differs only slightly from a Newtonian fluid). Equations (4) can be linearized in this case. Transforming in (4) and (5) to the dimensionless variables $\bar{r} = r/a$, $\bar{v}_r = v_r/aq$, $\bar{v}_\theta = v_\theta/aq$, $\bar{p} = p/p_\infty$ (p_∞ is the freestream pressure), $\bar{\psi} = \psi/a^3q$, $\bar{I} = I/3q^2$, we look for a solution of problem (4)-(5) in the form of asymptotic expansions in powers of the small parameter $\varepsilon = (n-1)/2$:

$$\begin{aligned} \bar{\psi} &= \bar{\psi}_0 + \varepsilon\bar{\psi}_1 + \varepsilon^2\bar{\psi}_2 + \ldots, \\ \bar{p} &= \bar{p}_0 + \varepsilon\bar{p}_1 + \varepsilon^2\bar{p}_2 + \ldots, \end{aligned}$$

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